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Hadronic Atoms in QCD

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Abstract

We propose a non relativistic effective Lagrangian approach to study hadronic atom observables in the framework of QCD (including photons). We apply our formalism to derive a general expression for the width of the $\pi^+\pi^-$ atom decaying into two neutral pions. It contains all terms at leading and next-to-leading order in isospin breaking. The result allows one to evaluate the combination $a_0 - a_2$ of $\pi\pi$ S-wave scattering lengths from $\pi^+\pi^-$ lifetime measurements, like the one presently performed by the DIRAC experiment at CERN.

The DIRAC collaboration at CERN [1] aims to measure the lifetime of the $\pi^+\pi^-$ atom (pionium) in its ground state at the 10% level. This atom decays predominantly into two neutral pions, $\Gamma = \Gamma_{2\pi^0} + \Gamma_{2\gamma} + \ldots$, with $\Gamma_{2\gamma}/\Gamma_{2\pi^0} \sim 4 \cdot 10^{-3}$ [1]. The measurement of $\Gamma_{2\pi^0}$ allows one [2]-[6] to determine the difference $a_0 - a_2$ of the strong S-wave $\pi\pi$ scattering lengths with isospin $I = 0, 2$. One may then confront the predictions for this quantity obtained in standard ChPT [7, 8] with the lifetime measurement, and furthermore analyze the nature of spontaneous chiral symmetry breaking in QCD [9]. In order to perform these investigations, one needs to know the theoretical expression for the width of pionium with a precision that properly matches the accuracy of the lifetime measurement of DIRAC. It is the aim of the present talk to derive a general formula [2] for the $\pi^+\pi^-$ atom decay width in the framework of QCD (including photons) by use of effective field theory techniques. Our result contains all terms at leading and next-to-leading order in the isospin breaking parameters $\alpha \simeq 1/137$ and $(m_u - m_d)^2$. On the other hand, we expect that the contributions from next-to-next-to-leading order are completely negligible, at least for the analysis of DIRAC data, and we therefore discard them here.

In several recent publications [10], the decay of $\pi^+\pi^-$ atoms has been studied in the framework of a non relativistic effective Lagrangian approach - a method originally proposed

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by Caswell and Lepage [11] to investigate bound states in general. This method has proven to be far more efficient for the treatment of loosely bound systems - such as the $\pi^+\pi^-$ atom - than conventional approaches based on relativistic bound-state equations. It allows one e.g. to go beyond the local approximation used in [12, 13]. On the other hand, we are not aware of a systematic investigation of the decay of the $\pi^+\pi^-$ atom in this framework. In particular, the chiral expansion of the width has not been discussed, and a comparison of the corrections found in this framework with the results of [12, 13] has never been provided. The main purpose of my talk - based on the papers [2, 14] - is to fill this gap.

The formation of the atom and its subsequent decay into two neutral pions is induced by isospin breaking effects in the underlying theory. In the present framework, these are the electromagnetic interactions and the mass difference of the up and down quarks. In the following, it is useful to count $\alpha \simeq 1/137$ and $(m_d - m_u)^2$ as small parameters of order δ. More than forty years ago, Deser *et al.* [4] derived the formula for the width of the $\pi^- p$ atom at leading order in isospin symmetry breaking. Later in Refs. [5, 6], this result was adapted to the $\pi^+\pi^-$ atom[2]. In particular, it was shown that - again at leading order in isospin symmetry breaking effects - the width $\Gamma^{\rm LO}_{2\pi^0}$ of pionium is proportional to the square of the difference $a_0 - a_2$,

$$\Gamma^{\rm LO}_{2\pi^0} = \frac{2}{9}\,\alpha^3 p^\star (a_0 - a_2)^2 \; ; \qquad p^\star = (M_{\pi^+}^2 - M_{\pi^0}^2 - \frac{1}{4} M_{\pi^+}^2 \alpha^2)^{1/2}. \tag{1}$$

At leading order in δ, the momentum $p^\star$ becomes $\sqrt{2M_{\pi^+}(M_{\pi^+} - M_{\pi^0})}$ - this is the expression used in [5, 6]. We prefer to use Eq. (1), because in this manner, one disentangles the kinematical corrections - due to the expansion of the square root - from true dynamical ones. In our recent article [2], we derived a general expression for the pionium lifetime, that is valid at leading and next-to-leading order in isospin breaking,

$$\Gamma_{2\pi^0} = \frac{2}{9}\,\alpha^3 p^\star \mathcal{A}^2 (1 + K). \tag{2}$$

The quantities $\mathcal{A}$ and K are expanded in powers of δ. In particular, it has been shown in [2] that[3]

$$\mathcal{A} \;\; = \;\; -\frac{3}{32\pi}\,{\rm Re} A^{+-00}_{\rm thr} + o(\delta), \tag{3}$$

where ${\rm Re} A^{+-00}_{\rm thr}$ is calculated as follows. One evaluates the relativistic scattering amplitude for the process $\pi^+\pi^- \to \pi^0\pi^0$ at order δ near threshold and removes the (divergent) Coulomb phase. The real part of this matrix element develops singularities that behave like $|\mathbf{p}|^{-1}$ and $\ln 2|\mathbf{p}|/M_{\pi^+}$ near threshold ($\mathbf{p}$ denotes the center of mass momentum of the charged pions). The remainder, evaluated at the $\pi^+\pi^-$ threshold $\mathbf{p} = \mathbf{0}$, equals ${\rm Re} A^{+-00}_{\rm thr}$. It contains terms of order δ^0 and δ and is normalized such that, in the isospin symmetry limit, $\mathcal{A} = a_0 - a_2$.

[2]There are a few misprints in Eq. (6) for the pionium decay rate in Ref. [5]. The correct result is displayed in Ref. [6].

[3]We use throughout the Landau symbols $O(x)$ [$o(x)$] for quantities that vanish like x [faster than x] when x tends to zero. Furthermore, it is understood that this holds modulo logarithmic terms, i.e. we write also $O(x)$ for $x \ln x$.

Finally, the quantity K starts at order $\alpha \ln \alpha$ - its explicit expression up to and including terms of order δ is given by

$$K = \frac{\Delta_\pi}{9M_{\pi^+}^2}\,(a_0+2a_2)^2 - \frac{2\alpha}{3}\,(\ln\alpha - 1)\,(2a_0+a_2) + o(\delta), \qquad \Delta_\pi = M_{\pi^+}^2 - M_{\pi^0}^2. \tag{4}$$

In the derivation of Eqs. (2) - (4), the chiral expansion has not been used. It is the aim of my talk to show how this result can be derived (details will be provided in a forthcoming publication [14]).

We proceed as follows. First, we display the non relativistic effective Lagrangian for pions, as derived from ChPT. Next, we formulate resonance two-channel $\pi\pi$ scattering theory by applying Feshbach's projection technique [15]. This method allows one to explicitly reveal the pole structure of the scattering matrix element and to obtain the equation for the bound-state energy of the $\pi^+\pi^-$ atom. Solving this equation, bound state characteristic of the hadronic atoms are given in terms of the couplings in the non relativistic Lagrangian. At the final stage, unknown couplings in the non relativistic Lagrangian can be expressed in terms of relativistic $\pi\pi$ scattering amplitudes through the matching procedure. Finally, as an illustration of our method, we derive Eqs. (2) - (4).

The non relativistic effective Lagrangian $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_D + \mathcal{L}_C + \mathcal{L}_S$ - at the order of accuracy we are working here - consists of the free Lagrangian for charged and neutral pions ($\mathcal{L}_0$), the "disconnected" piece ($\mathcal{L}_D$) - providing the correct relativistic relation between the energies and momenta of the pions - the Coulomb interaction piece ($\mathcal{L}_C$), and the "connected" piece ($\mathcal{L}_S$) which contains local four-pion interaction vertices:

$$\begin{aligned}
\mathcal{L}_0 &= \sum_{i=\pm,0} \pi_i^\dagger \Big(i\partial_t - M_{\pi_i} + \frac{\triangle}{2M_{\pi_i}} \Big) \pi_i, \qquad (5)\\
\mathcal{L}_D &= \sum_{i=\pm,0} \pi_i^\dagger \Big(\frac{\triangle^2}{8M_{\pi_i}^3} + \cdots \Big) \pi_i, \quad \mathcal{L}_C = -4\pi\alpha (\pi_-^\dagger \pi_-) \triangle^{-1} (\pi_+^\dagger \pi_+) + \cdots,\\
\mathcal{L}_S &= c_1 \pi_+^\dagger \pi_-^\dagger \pi_+ \pi_- + c_2 [\pi_+^\dagger \pi_-^\dagger (\pi_0)^2 + \text{h.c.}] + c_3\, (\pi_0^\dagger \pi_0)^2\\
&+ c_4 [\pi_+^\dagger \overset{\leftrightarrow}{\triangle} \pi_-^\dagger (\pi_0)^2 + \pi_+^\dagger \pi_-^\dagger \pi_0 \overset{\leftrightarrow}{\triangle} \pi_0 + \text{h.c.}] + \cdots,
\end{aligned}$$

where $u \overset{\leftrightarrow}{\triangle} v \equiv u\triangle v + v \triangle u$. The coupling constants c_i are real at $O(\alpha)$ and are determined through matching to the relativistic theory.

We now formulate the two-channel $\pi\pi$ scattering theory. We denote the full Hamiltonian derived from (5) by $H = H_0 + H_C + V$, with $V = H_D + H_S$. The scattering operator T obeys the Lippmann-Schwinger equation $T(z) = (H_C + V) + (H_C + V) G_0(z) T(z)$. The free and the Coulomb Green operators are defined as $G_0(z) = (z - H_0)^{-1}$ and $G(z) = (z - H_0 - H_C)^{-1}$, respectively. The pole structure of the T-matrix is predominantly determined by the static Coulomb interaction H_C, whereas V generates a small shift of the pole positions into the complex z-plane and will be treated perturbatively. To this end, we use the method developed by Feshbach [15] a long time ago. The T-matrix in our theory describes the transitions between charged $|\mathbf{P},\mathbf{p}\rangle_+ = a_+^\dagger(\mathbf{p}_1) a_-^\dagger(\mathbf{p}_2)|0\rangle$ and neutral $|\mathbf{P},\mathbf{p}\rangle_0 = a_0^\dagger(\mathbf{p}_1) a_0^\dagger(\mathbf{p}_2)|0\rangle$ states, where $a_i^\dagger$ denote the creation operators for non relativistic pions. Further, $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$ and $\mathbf{p} = \frac{1}{2}(\mathbf{p}_1 - \mathbf{p}_2)$ are the CM and relative momenta of pion pairs, respectively. We work in the

CM system and remove the CM momentum from the matrix elements of any operator R, introducing the notation ${}_A\langle \mathbf{P}, \mathbf{q}|R(z)|\mathbf{0}, \mathbf{p}\rangle_B = (2\pi)^3\delta^3(\mathbf{P})(\mathbf{q}|r_{AB}(z)|\mathbf{p})$, where $A,\ B = +,\ 0$. The operators $r_{AB}(z)$ act in the Hilbert space of vectors $|\mathbf{p})$, where the scalar product is defined as the integral over the relative three-momenta of pion pairs.

In order to avoid the complications associated with charged particles in the final states, we consider the elastic scattering process $\pi^0\pi^0 \to \pi^0\pi^0$. In the vicinity of the $\pi^+\pi^-$ threshold, the scattering matrix element develops a pole at [15]

$$z - E_0 - (\Psi_0|\tau_{++}(z)|\Psi_0) = 0, \tag{6}$$

where $(\mathbf{p}|\Psi_0) = \Psi_0(\mathbf{p})$ stands for the unperturbed Coulomb ground-state wave function, and E_0 is the corresponding ground-state energy. According to the conventional definition, the decay width is $\Gamma = -2\mathrm{Im}z$. The operator $\tau_{AB}(z)$ denotes the "Coulomb-pole removed" transition operator that satisfies the equation

$$\tau_{AB}(z) = v_{AB} + v_{A+}\hat{g}_{++}(z)\tau_{+B}(z) + \frac{1}{4}v_{A0}g_{00}(z)\tau_{0B}(z),$$

$$(\mathbf{q}|\hat{g}_{++}(z)|\mathbf{p}) = (\mathbf{q}|g_{++}(z)|\mathbf{p}) - \frac{\Psi_0(\mathbf{q})\Psi_0(\mathbf{p})}{(z - E_0)}\,. \tag{7}$$

It remains to solve Eq. (6) in the dimensional regularization scheme and with the use of the effective potential technique (see details in [2, 14]). We find that the width of the $\pi^+\pi^-$ atom - in terms of the effective couplings c_i - is given by

$$\Gamma_{2\pi^0} = \frac{\alpha^3 M_{\pi^+}^3}{8\pi^2}\rho^{1/2}M_{\pi^0}\Big(1+\frac{5\rho}{8M_{\pi^0}^2}\Big)(c_2-2\rho c_4)^2\Big(1-\rho c_3^2\frac{M_{\pi^0}^2}{4\pi^2}\Big)\Big(1-\frac{\alpha M_{\pi^+}^2}{4\pi}\xi c_1\Big)+\cdots, \tag{8}$$

where $\rho = 2M_{\pi^0}(M_{\pi^+} - M_{\pi^0} - M_{\pi^+}\alpha^2/8)$, $\xi = 2\ln\alpha - 3 + \Lambda + \ln(M_{\pi^+}^2/\mu^2)$,
$\Lambda = (\mu^2)^{d-3}[(d-3)^{-1} - \Gamma'(1) - \ln 4\pi]$. The ellipsis denotes higher order terms in isospin breaking. The divergent term proportional to Λ stems from a charged pion loop with one Coulomb photon exchange. It is removed by the renormalization procedure in the scattering sector. Next, we consider the matching procedure, which relates the effective couplings c_i to the $\pi\pi$ scattering amplitudes for three channels $\pi^+\pi^- \to \pi^0\pi^0$, $\pi^+\pi^- \to \pi^+\pi^-$ and $\pi^0\pi^0 \to \pi^0\pi^0$ evaluated in the relativistic theory [16]:

$$3M_{\pi^+}^2\, c_1 = 4\pi\,(2a_0 + a_2) + o(\delta), \qquad 3M_{\pi^+}^2\, c_3 = 2\pi\,(a_0 + 2a_2) + o(\delta), \tag{9}$$

$$\mathcal{A} = -\frac{3}{8\pi}M_{\pi^+}^2\Big[2c_2 - 4\Delta_\pi\Big(c_4 + \frac{c_2c_3^2}{8\pi^2}M_{\pi^0}^2\Big) + \frac{\alpha M_{\pi^+}^2}{4\pi}\Big(1 - \Lambda - \ln\frac{M_{\pi^+}^2}{\mu^2}\Big)c_1c_2\Big] + o(\delta) \tag{10}$$

Substituting Eqs. (9), (10) into Eq. (8), we finally arrive at the general formula for the $\pi^+\pi^-$ atom decay width in QCD, given by Eqs. (2) - (4). It is our opinion that these equations finalize the attempts to calculate the width $\Gamma_{2\pi^0}$ at next-to-leading order, relegating the problem to the evaluation of the physical on-mass-shell scattering amplitude for the process $\pi^+\pi^- \to \pi^0\pi^0$ to any desired order in the chiral expansion. Numerical analysis of the $\pi^+\pi^-$ atom lifetime in ChPT at one loop, including a comparison with recent work in literature, was performed recently in Ref. [3].

In conclusion, we have evaluated the width $\Gamma_{2\pi^0}$ of the $\pi^+\pi^-$ atom in its ground state at leading and next-to-leading order in isospin breaking. The non relativistic effective Lagrangian approach of Caswell and Lepage [11] appears to be an extremely suitable tool for this purpose, that allows one to completely solve this problem. Its usefulness may be seen even more clearly for the case of $p\pi^-$, pK^-, $d\pi^-$, dK^- atoms, studied in ongoing or planned experiments (PSI, KEK, DAΦNE), because this approach trivializes the spin-dependent part of the problem.

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